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Trace formulas for arithmetical systems

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For quantum problems on the pseudosphere generated by arithmetic groups there exist special trace formulas, called trace formulas for Hecke operators, which permit the reconstruction of wave functions from the knowledge of periodic orbits. After a short discussion of this subject we present the Hecke operators' trace formulas for the Dirichlet problem on the modular billiard, which is a prototype of arithmetical systems. The results of numerical computations for these semiclassical-type relations are in good agreement with the directly computed eigenfunctions.

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The problem of semiclassical quantization of ergodic systems has attracted wide attention in the last few years (see, e.g., [1,2] and references therein). The main tool here is the trace formula [3], which gives a connection between the density of quantum energy levels and a sum over all classical periodic trajectories of the system.

In general, this relation is valid only in the limit of the Planck constant tending to zero. The only exception, where such formulas are exact, is the problem of finding the spectrum of the Laplace-Beltrami operator on surfaces of constant negative curvature generated by discrete groups (see, e.g., [15,4]). In these cases there exists the famous Selberg trace formula [5,6], giving an exact relationship between the quantum spectrum and classical periodic orbits. The numerical computations for different models [7–11] have confirmed that such formulas are not merely of a purely theoretical interest.

The purpose of this Rapid Communication is to emphasize that for specific subclasses of models on constant negative curvature surfaces, namely, for ones generated by arithmetic groups, there exists another type of trace formula; it permits one (in principle) to obtain not only the energy eigenvalues but also the corresponding eigenfunctions directly from periodic orbits. Though in general one can build a semiclassical expression for wave functions through periodic orbits [12], the formula discussed below is of quite a different origin and seems to be not easily generalized for other systems. Nevertheless, it is important for semiclassical computations to investigate how classical periodic orbits conspire to reproduce quantum eigenfunctions.

Arithmetic groups are a specific subclass of discrete groups. We shall not give here the precise mathematical definitions; they can be found, e.g., in [13,6,14]. One can say as a crude analogy that arithmetic groups are among discrete groups as integers among rational numbers. The simplest and the most investigated example of arithmetic groups is the modular group (and its subgroups) (see, e.g., [15]), which is defined as the group of all 2×2 matrices where all entries are integers and the determinant is 1.

The peculiarities of quantum problems for arithmetic groups were stressed in [16], where it was shown that the arithmetic nature of these groups leads to exponentially large degeneracies of periodic orbits and to nonuniversal energy-level statistics, contrary to what was expected for ergodic systems [17].

Here we shall explore another property of such systems, namely, the existence of infinitely many operators commuting with the quantum Hamiltonian (and with themselves) (see also [18]). These operators are called Hecke operators and are of a purely arithmetical origin [14,15].

For simplicity we consider (as is usual in this subject) the case of the modular group, although most formulas could be generalized for other arithmetic groups [14].

Let us consider the set of 2×2 matrices, with integer entries as for the modular domain, but with the determinant being a certain integer p ($p\neq 1$, $p\neq 0$):

$$M_p=\left\{\begin{bmatrix} a & b \\ c & d\end{bmatrix},\ a,b,c,d \text{ are integers, } ad-bc=p\right\}. \tag{1}$$

Their importance comes from the (easily checked) fact that different matrices of the modular group could be conjugated by these matrices, even though they do not form a group, not being stable by multiplication.

An arbitrary matrix G of this form can be uniquely represented by the product [14,15]

$$G=g\alpha_p\ , \tag{2}$$

where g belongs to the modular group and α_p is one of the following fixed matrices:

$$\alpha_p=\begin{bmatrix} a & b \\ 0 & d\end{bmatrix},\ a,b,d \text{ integers, } ad=p\ ,\quad d>0,\ \ 0\le b\le d-1\ . \tag{3}$$

Let $\psi(x,y)$ be an automorphic function of the modular group [15]; i.e., it will obey $\psi(g(z))=\psi(z)$ for all matrices g from the modular group.

It is easy to see [15] that the function

$$\phi(z)=(T_p\psi)(z)=\frac{1}{p^{1/2}}\sum_{a,b,d}\psi\left(\frac{az+b}{d}\right), \tag{4}$$

where the summation is done over all a,b,d, as in Eq. (3), will also be an automorphic function for the modular group. This is a kind of symmetrization of $\psi(z)$ over the images of z by the elements of M_p.

The operators T_p defined in (4) are called Hecke operators. They form a commutative algebra and commute also with the Laplace-Beltrami operator [14,15]. Therefore, if there is no degeneracy of the eigenvalues of the Laplace-Beltrami operator, one has

$$T_p\psi_n(x,y)=c_p(n)\psi_n(x,y)\ , \tag{5}$$

i.e., an eigenfunction of the Laplace-Beltrami operator will be simultaneously an eigenfunction for all the Hecke operators.

Any eigenfunction of the Laplace-Beltrami operator for the modular group corresponding to the discrete spectrum can be written as the following Fourier decomposition (see, e.g., [15]):

$$\psi_n(x,y)=y^{1/2}\sum_{p=1}^{\infty}c_p(n)K_{s-1/2}(2\pi py)e^{2\pi ipx}\ , \tag{6}$$

where s is connected with a Laplace-Beltrami eigenvalue λ by the relation $\lambda=s(s-1)$ and $K_\nu(x)$ is the Hankel function.

Using the properties of the Hecke operators, one can show [15] (assuming nondegeneracies of the eigenvalues of the Laplace-Beltrami operator) that $c_p(n)$ in Eq. (5) coincide with $c_p(n)$ in Eq. (6), i.e., the eigenvalues of the pth Hecke operator are connected with the pth Fourier coefficients of the expansions of the eigenfunctions of the Laplace-Beltrami operator. This property is of particular importance because then the knowledge of the eigenvalues of all Hecke operators permits us to reconstruct wave functions.

To compute the trace formula for Hecke operators one has to consider the following sum:

$$\int_F\sum_{G\in M_p}k(z,Gz)d\mu(z)\ , \tag{7}$$

where the kernel $k(z,z')$ depends only on the hyperbolic distance between z and z', and F is the fundamental domain of the modular group. Then one regroups terms into conjugacy classes with respect to the group SL(2,Z). In [19] this trace formula was presented for p prime, and $p>0$. We found that for numerical computations it is more convenient to use a different trace formula corresponding to the Hecke operator applied to the Dirichlet kernel on a billiard defined in half the modular domain. More precisely, instead of applying T_p to $k(z,z')$ as in (7), one applies $\frac{1}{2}(T_p-T_{-p})$; as shown in [20] this gives a trace formula for a billiard with Dirichlet boundary conditions on half the modular domain (which is called the Artin billiard [11]). The group Γ with respect to which we compute the conjugacy classes is the full billiard modular group (BMG), i.e., the set of all 2×2 matrices with integer entries and determinant ± 1.

Though all ideas are not new [6,20] the computation is quite tedious and this formula to our knowledge has not been published before. We present here the final result of this computation; details will be published elsewhere [21]:

$$\sum_{n=0}^{\infty}c_p(n)h(r_n)=\frac{1}{\sqrt{p}}\sum_{\text{hyperbolic}}\frac{l_p}{2\sinh(L_p/2)}g(L_p)-\frac{1}{\sqrt{p}}\sum_{\text{hyperbolic}}\frac{l_{-p}}{2\cosh(L_{-p}/2)}g(L_{-p})$$
$$+\frac{1}{2\sqrt{p}}\sum_{\text{elliptic}}\frac{1}{2m\sin\theta}\int_{-\infty}^{+\infty}\frac{e^{-2\theta r}}{1+e^{-2\pi r}}h(r)dr+g(\ln p)\left[\ln\left(\frac{p-1}{p+1}\right)-\frac{\ln X(p-1)}{p-1}+\frac{\ln X(p+1)}{p+1}\right]$$
$$+\int_{\ln p}^{\infty}g(u)\frac{du}{e^{u/2}p^{1/2}-e^{-u/2}p^{-1/2}}\ . \tag{8}$$

Here $h(r)$ is any analytic function in the strip $|\mathrm{Im}r|\le\frac{1}{2}+\delta$, such that $h(-r)=h(r)$ and $|h(r)|<A|1+|r||^{-2-\delta}$, $A>0$, $\delta>0$.

$g(u)=(1/2\pi)\int_{-\infty}^{+\infty}h(r)e^{-iru}dr$ is the Fourier transform of $h(r)$. The summation on the right-hand side (rhs) is taken over all eigenvalues of the Laplace-Beltrami operator $\lambda_n=\frac{1}{4}+r_n^2$, and the $c_p(n)$ are the eigenvalues of the T_p Hecke operator. (We stress that there exists a trace formula for each p.) $X(n)=\prod_{k\ \mathrm{mod}\,n}(k,n)$, (k,n) being the greatest common divisor of k and n.

The first term corresponds to the matrices in M_p whose trace is greater than $2\sqrt{p}$ and that does not belong to a conjugacy class of some α_p; equivalently, those are matrices whose traces are greater than $2\sqrt{p}$ but not equal to $(p+1)$.

The corresponding term in the usual trace formula involves hyperbolic conjugacy classes, which are in one-to-one correspondence with the periodic orbits of the system. Here we do the same, taking a representative G for each conjugacy class with respect to the BMG in M_p. We can associate a length L_p with G, as with every hyperbolic fractional transformation, by

$$2\cosh\frac{L_p}{2}=\left|\mathrm{Tr}\frac{G}{\sqrt{p}}\right| . \tag{9}$$

Then one computes the *commutant* of it; i.e., matrices g of the full BMG which commute with G: $Gg=gG$. This set is, as previously, generated by a single element of the BMG whose associated periodic orbit has a length l_p. The formula for hyperbolic classes looks the same as in the usual case, but in the usual case $L_p=nl_p$, n being an integer, whereas here there is no simple relation between them. In fact, our results show that in some cases l_p can be enormously greater than L_p. In other words, G as a transformation maps one point on the periodic orbit of length l_p into another point on the same periodic orbit, which is not connected in a simple way to the starting point. We can say that those L_p of hyperbolic classes of M_p are the hyperbolic distances between a point and its image.

The second term gives the same as above for $-p$; the matrices of M_{-p} have a negative determinant and correspond to odd boosts. Here, as above, L_{-p} is the length associated to a hyperbolic class in M_{-p} with respect to the BMG:

$$2\sinh\frac{L_{-p}}{2}=\left|\mathrm{Tr}\frac{G}{\sqrt{p}}\right| ; \tag{10}$$

the trace of the matrices in M_{-p} should be different from $(p-1)$. l_{-p} is the length of the generator of the commutant in the BMG and corresponds to the length of a periodic orbit of the billiard.

The contribution of matrices of M_{-p} with trace equal to zero have an additional factor $\frac{1}{2}$ due to the existence of a commuting element whose square is the identity.

The third term corresponds to the conjugacy classes of elliptic matrices G in M_p, i.e., whose trace is less than $2\sqrt{p}$ (there are no such classes in M_{-p}); then we can write $\mathrm{Tr}(G)=2\sqrt{p}\cos\theta$ with $0<\theta<\pi$. The only possible matrices of the billiard modular group commuting with those matrices are elliptic matrices corresponding to a primitive rotation of angle $2\pi/m$. It is this integer m that enters Eq. (8) ($m=1$ corresponds to the identity matrix).

The last two terms are connected with the existence of the infinite cusp, which usually leads to difficulties in deriving trace formulas for the modular group [6,19,20]. They correspond to the conjugacy classes of the matrices α_p and α_{-p} of Eq. (3). The commutant of those matrices is trivial, being merely the identity matrix. We note that two different α_p can belong to the same BMG conjugacy class.

We can interpret this formula as the usual trace formula applied to a system with symmetry by seeing which periodic orbits of the BMG will appear in Eq. (8). The only ones that will be selected will be those whose corresponding matrix T commutes with one matrix G of M_p (or M_{-p}). From Eq. (2) one knows that G can be written $G=g\alpha_p$, g being in $SL(2,Z)$ and α_p having the form (3). It is easy to show that if T commute with $g\alpha_p$, then $g^{-1}Tg=\alpha_pT\alpha_p^{-1}$.

This means that $\alpha_pT\alpha_p^{-1}$ is a matrix of the BMG belonging to the same conjugacy class as T; then $\alpha_pT\alpha_p^{-1}$ corresponds to a periodic orbit of the Artin billiard and this periodic orbit is the one given by T. So the periodic orbits of the Artin billiard that are selected are those that are invariant by the action of α_p. For example, T_{-1} acts as the symmetry with respect to the axis $x=0$, and the corresponding trace formula for $\frac{1}{2}(T_1-T_{-1})$ gives the usual trace formula for the Dirichlet problem on the Artin billiard [20].

We have computed the rhs of Eq. (8) for a few values of p. We chose as the function $h(r)$ a Gaussian function $h(r)=\exp[-A(r-r_0)^2]+\exp[-A(r+r_0)^2]$, for which the rhs of the trace formula (8), considered as the function of k_0, should have peaks at true eigenvalues, the amplitudes of which are equal to the Fourier coefficients of the expansion (6). For each value of the trace of M_p we computed the representatives of conjugacy classes of M_p with respect to BMG by using a generalization of the well-known Gauss method of reduction of quadratic forms, and then from the knowledge of the intermediate steps of this procedure we computed the matrix of the BMG of minimal length that commute with it; this matrix is the generator of the commutant. We have found for some of the M_p matrices an unexpectedly large length for the commuting matrices. For example, the matrix with entries 295, 274; 267, 248 has a determinant equal to 2 and the matrix of the BMG of minimal trace that commutes with it has integer entries on the order of 10^{471}. The details of the method used will be given elsewhere [21].

The hyperbolic terms give the main contribution to the formula (8). The elliptic ones are exponentially small at the energies of computation, and the "cusp" terms give more or less a smooth slowly oscillating function of period $2\pi/\ln p$ with small amplitude.

We have computed the sum on the rhs of Eq. (8), using approximately 15 000 periodic orbits, and have compared it with the results of direct computations of the Fourier coefficients and eigenvalues for this problem. As an example we presented in Figs. 1 and 2 the results of the computations for $p=3$, $A=30$ and for $p=5$, $A=28$. The classical and quantum computations seem almost indistinguishable at the scale used. The results for other values of p are of the same quality. For the first eigenvalues our computations are in perfect agreement with the results of Hejhal [22]. So this method seems to be efficient for exploring the eigenfunctions of arithmetical systems; it does not use a lot of computer time to get the

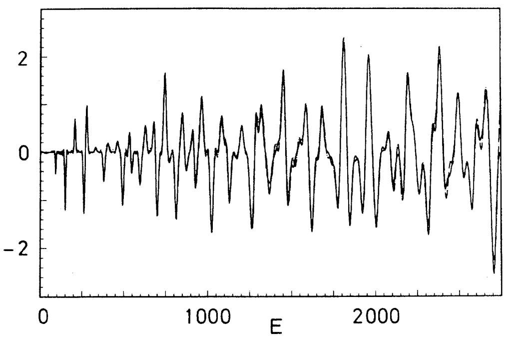

FIG. 1. The Hecke operator's trace formula for $p=3$; the dashed curve is our result with 15 000 periodic orbits, the continued one is the quantum calculation.

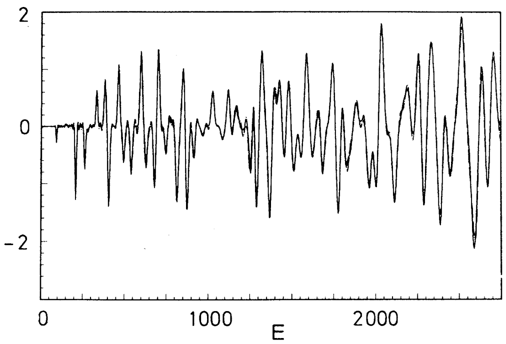

FIG. 2. The same as Fig. 1 but for $p=5$.

results we show.

We have also studied the number of conjugacy classes in M_p with respect to their length L_p for different p. It was found that this number grows more slowly than in usual hyperbolic systems; for these ones, Huber's law [23] states that the number of periodic orbits of length less than L is $N(l<L)\sim \exp L/L$. Our computation gives a dependence like

$$\ln N(L_p<L)\sim \tfrac{3}{4}L \ . \tag{11}$$

The coefficient $\frac{3}{4}$ is a numerical one and we cannot exclude a slow dependence of this factor on p.

In conclusion we emphasize the following points.

- The existence of an infinite number of commuting Hecke operators is the characteristic property of models generated by arithmetic groups.
- For each value of p there exists a subset of periodic orbits of the modular domain, each of which remains invariant under one of transformations (3). The number of these orbits grows approximately as in (11).
- It is these invariant periodic orbits that give the contributions to the trace formulas for Hecke operators. The latter are a new kind of trace formula that permits us to reconstruct quantum wave functions of arithmetical systems directly from classical periodic orbits.
- The Hecke-type trace formula for Artin's billiard was derived and checked numerically; a good agreement between quantum and classical calculations was found.

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